No-passing rule in the ground state evolution of the random-field Ising model

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We exactly prove the no-passing rule in the ground state evolution of the random-field Ising model with monotonically varying external field. In particular, we show that the application of the no-passing rule can speed up the calculation of the zero-temperature equilibrium $M(H)$ curve dramatically.

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I. INTRODUCTION

The no-passing rule was first introduced by Middleton in the study of sliding charge-density waves (CDW's) [1]. The CDW problem belongs to the more general class of driven elastic manifolds in random media. If one defines a simple one-dimensional order parameter within the model, then a *natural partial ordering* of the configurations can be defined. In the simple CDW model considered by Middleton, the CDW configuration $\{\varphi_i(t)\}$ describes the CDW distortions at N lattice sites indexed by i, with $\varphi_i(t)$ real phase variables and t the time. The equation of motion for an overdamped CDW is governed by the Langevin dynamics

$$\dot{\varphi}_i = \Delta^2 \varphi_i - V_i'(\varphi_i) + f(t). \tag{1}$$

Here, the $\Delta^2\varphi_i$ term represents the simple elastic interactions. $V_i'(\varphi_i)$ is the pinning force at site i due to the 2π periodic pinning potential $V(\varphi_i)$. And $f(t)$ stands for the external driving force. Then one can define the natural partial ordering of two configurations: $C^G=\{\varphi_i^G\} \geq C^L=\{\varphi_i^L\}$ if $\varphi_i^G \geq \varphi_i^L$ for each site i of the system. The no-passing rule states that given a driving force f if initially $C^G(0) \geq C^L(0)$, then $C^G(t) \geq C^L(t)$ for all $t>0$, i.e., the "greater" (C^G) is never passed by the "lesser" (C^L). As stressed by Middleton, this rule relies crucially on the elastic potential being *convex*. In other words, the elastic potential tends to decrease the separation of nearest-neighbor φ's. More recently, Krauth *et al.* found a similar no-passing rule in the study of driven elastic strings in disordered media [2,3]. Obviously, this is the same general problem. Again, the rule is crucially dependent on the fact that the elastic potential is convex.

The no-passing rule can be used to prove many useful properties, such as the asymptotic uniqueness of the sliding state for CDW's [1] and the intriguing memory effects [4]. In fact, just after its introduction by Middleton, the no-passing rule was used in the nonequilibrium zero-temperature random-field Ising model (RFIM) by Sethna *et al.* to prove the return point memory effect [4]. The RFIM is obtained by adding a random field h_i at each site of the Ising model

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J s_i s_j - \sum_i (H + h_i) s_i. \tag{2}$$

The distribution of h_i values is usually taken to be Gaussian, with standard deviation R and mean 0. R is often called the disorder parameter. J is the nearest-neighbor ferromagnetic coupling strength and H is the uniform external field. In this case, the natural partial ordering of two spin configurations can be defined similarly as in the CDW case. The no-passing rule states: Let a system $C^G(t)$ be evolved under the fields $H^G(t)$ and similarly $C^L(t)$ evolved under $H^L(t)$. Suppose the fields $H^G(t) \geq H^L(t)$ and the initial configurations satisfy $C^G(0) \geq C^L(0)$, then $C^G(t) \geq C^L(t)$ at all times later $t>0$, i.e., the partial ordering will be preserved by the dynamics. With a local metastable single-spin-flip dynamics, i.e., a spin flips when its effective local field

$$h_i^{\text{eff}} = J\sum_j s_j + h_i + H \tag{3}$$

changes sign, the proof of the no-passing rule is straightforward [4]. Even with a two-spin-flip dynamics, it has been shown by Vives *et al.* that the no-passing rule is still true at zero temperature [5]. Note that for the magnetization process, the no-passing rule is equivalent to the fact that the flipped spins can never flip back as H is swept monotonically. Again, this rule is not unconditionally true. It relies crucially on the nearest-neighbor interaction being *ferromagnetic* ($J>0$). Just like the convex elastic potential, the ferromagnetic interaction also tends to decrease the separation of nearest-neighbor degrees of freedom, i.e., it tends to align the spins.

Recently, in the study of the equilibrium zero-temperature RFIM, Vives *et al.* conjectured that when the external field H is swept from $-\infty$ to ∞, flipped spins cannot flip back in the equilibrium $M(H)$ curve [6]. In other words, the no-passing rule is valid even for the zero-temperature equilibrium dynamics, i.e., the evolution of the ground state (GS). Vives *et al.* further conjectured that this rule can be used to speed up the calculation of the equilibrium $M(H)$ curve since flipped spins at a lower field can be removed from the GS calculation for all higher fields. Unfortunately, this simple but powerful rule has not been proven so far for the equilibrium RFIM. This is the main motivation of our work.

This paper is organized as follows. In Sec. II, we give a short introduction to the calculation of the equilibrium $M(H)$ curve of the zero-temperature RFIM. In Sec. III, we work out some basic steps for the proof of the equilibrium no-passing rule. In Sec. IV, we present the proof. In Sec. V, we show the direct application of this rule to the calculation of the equilibrium $M(H)$ curve. Finally, in Sec. VI we discuss its validity in other systems.

II. EQUILIBRIUM $M(H)$ CURVE

To calculate the equilibrium $M(H)$ curve of the zero-temperature RFIM, we first need to calculate the exact GS in

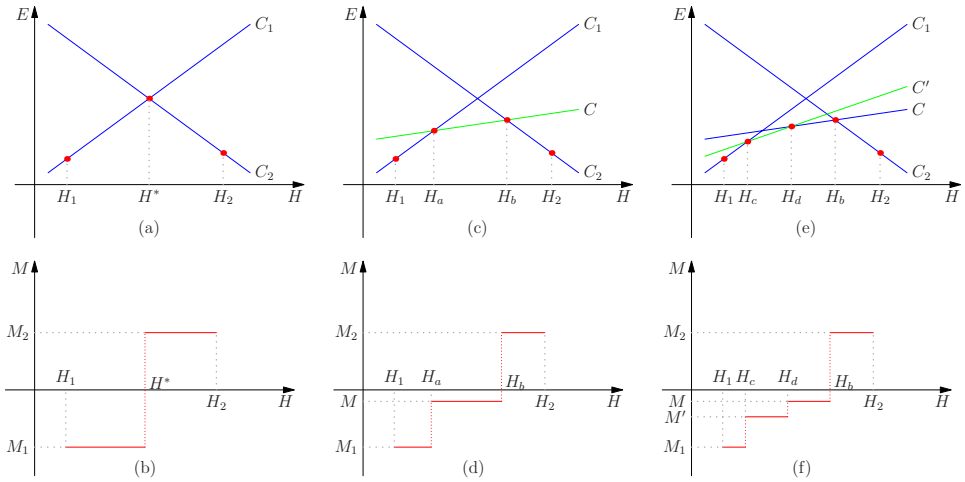

FIG. 1. (Color online) An illustration of the algorithm to calculate the equilibrium M-H curve. Calculate the energies E_1 and E_2 of the two simplest states $C_1=\{s_i=-1\}$ and $C_2=\{s_i=+1\}$, respectively, as a function of H. According to Proposition 1, C_1 (or C_2) would be the ground state for $H<-h_{\max}$ (or $H>-h_{\min}$). Calculate the crossing field $H^*(C_1,C_2)$ where $E_1=E_2$. Check whether there is a GS at H^*, which is different from C_1 and C_2. If no, the algorithm ends. If yes, denote the GS as C, calculate the crossing field $H^*(C,C_1)$ and $H^*(C,C_2)$. At the new crossing fields, check whether there is a GS which is different from the two intersected states. The algorithm will not end until all the crossing fields have been checked. An example of the calculated equilibrium M-H curve is shown in Fig. 2.

the RFIM at an arbitrary applied external field H. This is the basic step of calculating the equilibrium $M(H)$ curve, i.e., the GS evolution for varying H. Fortunately, there is a well-known mapping of the RFIM GS problem to a min-cut or max-flow problem in combinatorial optimization. The mapping and the so-called push-relabel algorithm for the min-cut or max-flow problem has been well described in the literatures [7,8]. For RFIM, the run time of the push-relabel algorithm scales as $\mathcal{O}(N^{4/3})$ with N the system size [9,10].

The equilibrium $M(H)$ curve can be simulated with the method reported in Refs. [9,11]. It is essentially based on the fact that the GS energy $E(\{s_i\},H)$ is convex up in H, which allows for estimates of the fields H where the magnetization jumps (called "avalanches" occur). This algorithm finds steps by narrowing down ranges where the magnetization jumps with an efficient linear interpolation scheme. An illustration of the algorithm is shown in Fig. 1. The details have been explained extensively in Ref. [11]. An example of the calculated equilibrium $M(H)$ curve is shown in Fig. 2.

In the E-H diagram, for each state $\{s_i\}$, the total energy E is represented by a straight line with slope $-M\equiv-\Sigma_i s_i$ since

$$E(\{s_i\},H)=E_0(\{s_i\})-HM, \tag{4}$$

with

$$E_0(\{s_i\})=-\sum_{\langle i,j\rangle}Js_is_j-\sum_i h_is_i \tag{5}$$

the energy axis intercept. E_0 is also called the *internal energy*, i.e., the total energy of the configuration when $H=0$. And the total energy E is also referred to as the *magnetic enthalpy*. Consider a D-dimensional hypercubic lattice of size $N=L^D$. Let $h_{\max}$ ($h_{\min}$) be the maximum (minimum) values of h_i for a certain realization of the random fields.

Four simple propositions follow here, which are very useful in understanding the algorithm to calculate the equilibrium M-H curve. Note that Propositions 1, 2, and 3 are just reproduced from Ref. [11]. The proofs have been given there. Proposition 4 is new and the proof is given in the Appendix.

Proposition 1. For $H<-h_{\max}$ ($H>-h_{\min}$), the ground state is $\{s_i=-1\}$ ($\{s_i=+1\}$).

Proposition 2. Let the spin configuration C_1 (C_2) be the ground state for $H=H_1$ ($H=H_2$). They have magnetization M_1 and M_2, respectively. If $C_1\neq C_2$ and $H_2>H_1$, then $M_2>M_1$.

Thus, when sweeping the external field from $H=-\infty$ to $H=\infty$, the magnetization M will increase monotonically. A corollary of this proposition is that in the $E-H$ diagram if the

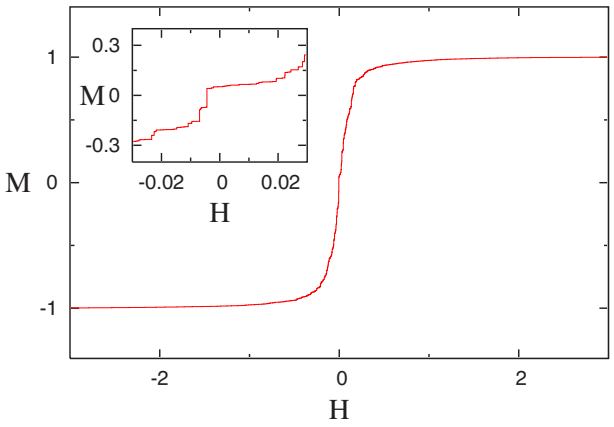

FIG. 2. (Color online) The equilibrium M-H curve (the ground state evolution) for the Gaussian RFIM with $D=3$, $L=32$, and $R=2.837$. Here R is the standard deviation of the Gaussian random-field distribution. The inset shows a detail of the $M(H)$ curve near $H=0$, where magnetization jumps are clearly seen.

slopes of the lines corresponding to the ground states C_1 and C_2 are different, i.e., $M_1 \neq M_2$, and without loss of generality we can assume $M_1 < M_2$, then the lines intersect at a field H^* such that $H_1 < H^* < H_2$. This field $H^*(C_1, C_2)$ is defined as the *crossing field* between C_1 and C_2. According to the definition, one has $E_0(C_1) - H^* M_1 = E_0(C_2) - H^* M_2$, so

$$H^*(C_1, C_2) = \frac{E_0(C_2) - E_0(C_1)}{M_2 - M_1}. \tag{6}$$

For example, we can calculate the crossing field between the two simplest ground states: $C_1 = \{s_i = -1\}$ with $M_1 = -N$ and $C_2 = \{s_i = +1\}$ with $M_2 = N$. We have $H^* = -1/N \Sigma_i h_i = -\bar{h}_i$ [12].

Proposition 3. Let the spin configuration C_1 (C_2) be the ground state for $H = H_1$ ($H = H_2$). $C_1 \neq C_2$, $H_2 > H_1$, and the crossing field between C_1 and C_2 is H^*. If there is no configuration C such that $E(C, H^*) < E(C_1, H^*) = E(C_2, H^*)$ then (i) C_1 is the ground state at least for the field range $[H_1, H^*)$ and (ii) C_2 is the ground state at least for the field range $(H^*, H_2]$.

This is the most important proposition. Its power comes from the fact that it can be applied iteratively. And finally, we get the $M(H)$ curve with all the ground states (see Fig. 1).

Proposition 4. If the ground state is nondegenerate, then there cannot be more than one avalanche connecting two nearest ground states in the E-H diagram.

The proof of this interesting proposition is shown in the Appendix. This proposition says that if the GS is nondegenerate, when we increase the external field H adiabatically slowly, we can trigger just one avalanche at a time.

III. PREPARATIONS OF THE PROOF

In this section, we will work out the total energy change of the spin configuration due to multiple spin flips and external field varying. The spin configuration is not necessarily the ground state.

First, let us consider the simplest case of a single spin flip. Suppose only one spin (s_i) flips during the evolution of configuration C at H to configuration C' at H', with $\Delta H = H' - H$ and $\Delta M = M' - M$. Define n_i (or n_i') to be the number of the ith spin's nearest neighbors that point in the same direction as the spin in configuration C (or C'). We call these spins the same-direction nearest neighbors (SDNN) of the ith spin. Note that $n_i = 0, 1, 2, \ldots, Z$ with $Z = 2D$ the coordination number of the D-dimensional hypercubic lattice.

It is easy to get the bond energy change $4J(n_i - D)$. The total energy change due to the single spin flip and the varying external field is given by

$$f_i(H, \Delta H) = f_i(H) - \Delta H M'. \tag{7}$$

Here we have defined

$$f_i(H) \equiv f_i(H, 0) = 4J(n_i - D) - (h_i + H)\Delta s_i, \tag{8}$$

which is the energy change due to spin i flipping for the configuration C just at the field H, i.e., $\Delta s_i = \pm 2$ with $\Delta H = 0$. It is easy to check that

$$f_{i,\pm}(H) = f_{i,\pm}(0) \pm 2H = \pm 2h_i^{\text{eff}}, \tag{9}$$

where "±" represents $s_i = \pm 1$ and $\Delta s_i = \mp 2$, accordingly.

Second, we consider two spin flips. Suppose two different spins (s_i and s_j) flip during the evolution of configuration C at H to configuration C' at H'. There are two subcases.

(1) s_i and s_j are not next to each other. The energy change is

$$f_{i,j}(H, \Delta H) = f_i(H) + f_j(H) - \Delta H M'. \tag{10}$$

(2) s_i and s_j are next to each other. The energy change is

$$f_{\langle i,j \rangle}(H, \Delta H) = f_i(H) + f_j(H) - 4J(s_i \cdot s_j) - \Delta H M'. \tag{11}$$

Note that the term $-4J(s_i s_j)$ is just due to the fact that the energy of the i-j bond will not change during the flip.

Finally, let us consider the general case [see Fig. 3(a)]. There are many spin flips during the evolution of configuration C at field H to C' at field H'. It is easy to check that the total energy change is given by

$$\Delta E(H, \Delta H) = [f_i(H) + f_j(H) + \cdots] - 4J(s_i s_j + \cdots) - \Delta H(M + \Delta s_i + \Delta s_j + \cdots). \tag{12}$$

On the right-hand side (RHS), the first term includes all the flipping spins. The second term includes all the nearest-neighbor interactions among those flipping spins. The last term is due to the varying external field. In particular, if all the flipping spins flip at the same H and they are connected to each other and have the same spin value -1 (or $+1$) before the flip, then this collective spin flip is called *an avalanche* (or *a reverse avalanche*).

Denote the energy change due to an avalanche A_α as $f_{A_\alpha}(H, \Delta H)$; we have

$$\begin{aligned} f_{A_\alpha}(H, \Delta H) &= [f_i(H) + f_j(H) + \cdots] - 4JN_b(A_\alpha) \\ &\quad - \Delta H(M + 2S_\alpha) \\ &\equiv f_{A_\alpha}(H) - \Delta H(M + 2S_\alpha), \end{aligned} \tag{13}$$

with $N_b(A_\alpha)$ defined as the number of interacting bonds in A_α, S_α the size of the avalanche, and $f_{A_\alpha}(H)$ the energy change due to the avalanche when $\Delta H = 0$. Similarly, for the reverse avalanche, we have

$$\begin{aligned} f_{A_\beta^r}(H, \Delta H) &= [f_i(H) + f_j(H) + \cdots] - 4JN_b(A_\beta^r) \\ &\quad - \Delta H(M - 2S_\beta^r) \\ &\equiv f_{A_\beta^r}(H) - \Delta H(M - 2S_\beta^r). \end{aligned} \tag{14}$$

Due to Eq. (9), we have

$$f_{A_\alpha}(H) = f_{A_\alpha}(0) - 2S_\alpha H, \tag{15}$$

$$f_{A_\beta^r}(H) = f_{A_\beta^r}(0) + 2S_\beta^r H. \tag{16}$$

Now we can rewrite Eq. (12) in terms of f_{A_α} and $f_{A_\beta^r}$. The total energy change due to avalanches and reverse avalanches is given by

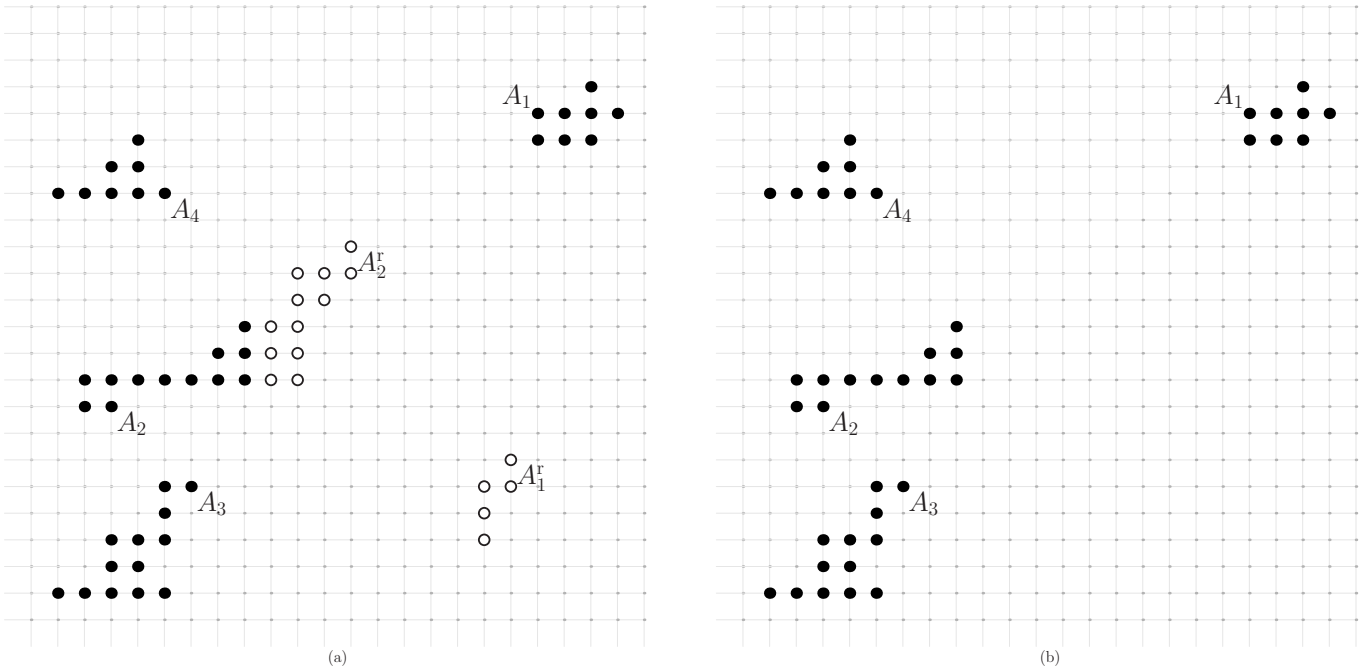

FIG. 3. States evolved from C_1, with only the change of the spin configuration, i.e., avalanches and reverse avalanches are explicitly shown. (a) State C_2: evolved from state C_1 with both avalanches and reverse avalanches. (Black dot) Spins flip up, forming avalanches (A_1, A_2, A_3, and A_4). (White dot) Spins flip down (reverse flip), forming reverse avalanches (A_1^r, A_2^r). Note that there are three interacting bonds between avalanche A_2 and reverse avalanche A_2^r. (b) State $\tilde{C}$ evolved from state C_1 without reverse avalanches.

$$\Delta E(H,\Delta H) = F_A(H) + F_{A^r}(H) + 4JN_b(A,A^r) - \Delta H(M + 2S_A - 2S_{A^r}), \tag{17}$$

with notations $F_A(H) \equiv \Sigma_\alpha f_{A_\alpha}(H)$, $F_{A^r}(H) \equiv \Sigma_\beta f_{A^r_\beta}(H)$, $S_A \equiv \Sigma_\alpha S_\alpha$, and $S_{A^r} \equiv \Sigma_\beta S^r_\beta$. Here $N_b(A,A^r)$ denotes the number of interacting bonds between avalanches and reverse avalanches. For example, in Fig. 3(a), $N_b(A,A^r)=3$.

IV. PROOF OF THE NO-PASSING RULE

Now we are ready for the proof of the no-passing rule. Let the spin configuration C_1 (C_2) be the ground state for $H=H_1$ ($H=H_2$), $H_2>H_1$. Suppose C_1 and C_2 are connected with multiple avalanches: $A_1, A_2, \ldots, A_n$ with sizes $S_1, S_2, \ldots, S_n$ and reverse avalanches $A_1^r, A_2^r, \ldots, A_m^r$, with sizes $S_1^r, S_2^r, \cdots, S_m^r$, respectively. To compensate these reverse avalanches (so as to make sure M is monotonically increasing, see Proposition 2), we must have $S_A = \Sigma_{\alpha=1}^n S_\alpha > S_{A^r} = \Sigma_{\beta=1}^m S^r_\beta$ [see Fig. 3(a)].

The idea is that if C_2 is the GS at field H_2, then it should have lower energy than any other spin configuration at H_2. But we will prove this is *not* true. Just consider another spin configuration $\tilde{C}$. The only difference between C_2 and $\tilde{C}$ is that $\tilde{C}$ is evolved from C_1 without any reverse avalanches [see Fig. 3(b)]. We now try to prove that $E(\tilde{C},H_2) < E(C_2,H_2)$, so C_2 cannot be the GS at H_2. But this is equivalent to proving that $\Delta\tilde{E} < \Delta E$. Here,

$$\begin{aligned}\Delta E &\equiv E(C_2,H_2) - E(C_1,H_1) \\ &= F_{A^r}(H_1) + F_A(H_1) + 4JN_b(A,A^r) \\ &\quad - \Delta H(M - 2S_{A^r} + 2S_A).\end{aligned} \tag{18}$$

On the other hand,

$$\Delta\tilde{E} \equiv E(\tilde{C},H_2) - E(C_1,H_1) = F_A(H_1) - \Delta H(M + 2S_A). \tag{19}$$

Therefore,

$$\Delta E - \Delta\tilde{E} = F_{A^r}(H_1) + 4JN_b(A,A^r) + 2S_{A^r}\Delta H > 0. \tag{20}$$

Here we have used the fact that C_1 is the ground state for $H=H_1$ such that any kinds of spin flip will increase the energy: $f_{A^r_\beta}(H_1) > 0 \Rightarrow F_{A^r}(H_1) > 0$. Also, for the ferromagnetic RFIM, $J>0$. Since each term is positive, so the sum is positive, i.e., $\Delta E > \Delta\tilde{E}$ or $E > \tilde{E}$. Actually, for any state C_2, which evolved from C_1 with reverse avalanches, we can find a corresponding state $\tilde{C}$, which evolved from C_1 without any reverse avalanches that has lower energy than C_2 at field H_2. So reverse spin flips are impossible for ground state evolution when increasing external field. Generally, flipped spins can never flip back when we sweep the external field monotonically.

V. APPLICATION

The straightforward application of the no-passing rule is very useful to accelerate the calculation of the ground states

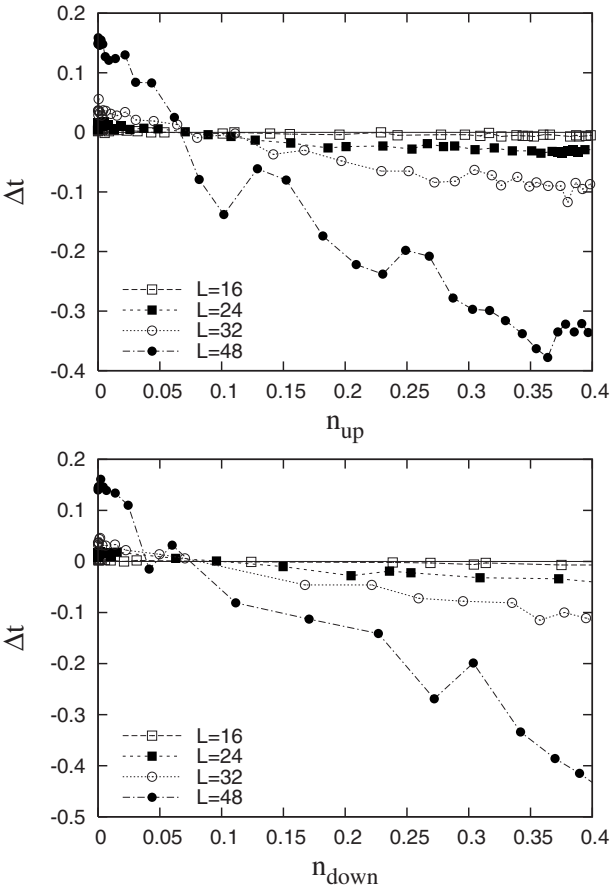

FIG. 4. Running time difference (Δt) between methods with and without using the earlier solution, i.e., the GS C_1 at field H_1, to calculate the GS C_2 at field H_2. The time difference Δt (given in seconds) is plotted against the up-spin (or down-spin) density of the GS C_1. $\Delta t<0$ means using the earlier solution will save the running time. Calculations are done for 3D Gaussian RFIM (with disorder parameter $R=2.27$) for different system sizes. (Top) $H_2>H_1$. Up spins in C_1 at field H_1 will stay up at field H_2. (Bottom) $H_2<H_1$. Down spins in C_1 at field H_1 will stay down at field H_2.

when varying the external field. Suppose the GS C_1 at field H_1 has already been obtained, and we want to calculate the GS C_2 at field H_2 with $H_2>H_1$. According to the no-passing rule, the up spins in C_1 will stay up in C_2, i.e., those spins are frozen, so we need not consider them in the ground state analysis. We just need to consider the down spins in C_1. The only cost is that we have to deal with the frozen up spins as complicated fixed boundary conditions for the down spins [13]. At first sight, one may think that only when the density of the frozen spins is big enough can we make the GS calculation faster. But how big is enough? To optimize our calculation, we consider the running time difference (Δt) between the two methods: (I) without using the earlier solution C_1, and (II) using the earlier solution C_1. For both methods, ground states are found by using the push-relabel algorithm. The numerical experiments are conducted on a desktop with 2.80 GHz CPU and 2GB memory. And we tune the up-spin density n_{up} (down-spin density n_{down}) by varying H_1. The result is shown in Fig. 4. It is found that for $H_2>H_1$, as long as $n_{\mathrm{up}}\gtrsim 0.07$ in GS C_1, method II will be faster than I. Symmetrically, for $H_2<H_1$, as long as $n_{\mathrm{down}}\gtrsim 0.07$ in GS C_1, method II will be faster than method I. This suggests it is not necessary to have an extremely large portion of frozen spins to use the earlier solution. Freezing a tiny part of spins will accelerate the GS calculation already. Furthermore, for larger and larger density of the frozen spins, using the earlier solution will save more and more running time. [Keep in mind that for RFIM, the running time of the push-relabel algorithm scales as $\mathcal{O}(N^{4/3})$.] Consequently, the calculation of the whole $M(H)$ curve will be sped up dramatically.

VI. DISCUSSIONS

Throughout our proof of the no-passing rule, we do not assume that the ground state is unique. In other words, the no-passing rule is correct even when the ground state is degenerate. For example, this happens for the RFIM when the random fields are chosen from a bimodal distribution [14].

In the proof we explicitly use the fact that the nearest-neighbor interaction should be *ferromagnetic* ($J>0$). This means any antiferromagnetic interactions will destroy the no-passing rule. Thus, for other random magnet models, if J_{ij} could be negative, such as the random-bond Ising model (RBIM) with negative J_{ij} or the spin glasses, the rule will be violated.

Finally, we conjecture that for elastic manifolds in random media, there could be a similar equilibrium no-passing rule at zero temperature, provided that the elastic potential is convex and partial ordering of the configurations can be clearly defined.

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APPENDIX: A SINGLE AVALANCHE CONNECTS TWO NEAREST GROUND STATES

Here, we show the proof of Proposition 4.

Proof. Suppose when the field is increased from H_1 to H_2, the GS C_1 evolves to the nearest GS C_2 with two avalanches (A_1 and A_2 with size $S_1\geq 1$ and $S_2\geq 1$, respectively).

The crossing field is given by

$$H^*(C_1,C_2)=\frac{E_0(C_2)-E_0(C_1)}{M_2-M_1}=\frac{f_{A_1}(0)+f_{A_2}(0)}{2(S_1+S_2)}. \quad \text{(A1)}$$

The last line is due to Eq. (17). We can choose a trial state C, which is evolved from C_1 with only avalanche A_1 occurring. We want the following relation to hold:

$$E(C,H^*)<E(C_1,H^*)=E(C_2,H^*), \quad \text{(A2)}$$

which is equivalent to

$$E(C,H^*) - E(C_1,H^*) = f_{A_1}(H^*) = f_{A_1}(0) - 2S_1H^* < 0. \tag{A3}$$

Plugging Eq. (A1) in it, we just need to prove

$$S_2 f_{A_1}(0) < S_1 f_{A_2}(0). \tag{A4}$$

If we do have $S_2 f_{A_1}(0) < S_1 f_{A_2}(0)$, then we choose the trial state C, which has lower energy than C_1 and C_2 at the field H^*; if $S_2 f_{A_1}(0) > S_1 f_{A_2}(0)$, then we can choose another trial state C', which is evolved from C_1 through only avalanche A_2 and has lower energy than both C_1 and C_2 at H^*. In both cases, we have shown that C_2 cannot be the nearest GS at H_2 for the GS C_1 at H_1, if C_1 evolves to C_2 with two avalanches. If $S_2 f_{A_1}(0) = S_1 f_{A_2}(0)$, it is easy to show that $E(C_1,H^*) = E(C,H^*) = E(C',H^*) = E(C_2,H^*)$, then there will be degenerate ground states at H^*, which is in contradiction to the hypothesis of Proposition 4.

Q.E.D.

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[12] Random fields are usually generated according to a zero-mean distribution. But in practical simulation we are always dealing with a finite size system. So it is impossible to get exactly $\bar{h}_i = 1/N\Sigma_i h_i = 0$.

[13] The mapping of the RFIM GS problem to a min-cut/max-flow problem has been well established. For details, see Ref. [7] (Chap. 6, Eqs. 6.12–6.15). If there are frozen spins, the problem might look somewhat complicated. However, by reorganizing the terms in the Hamiltonian, we find that the frozen spins just have two effects. First, they reduce the degrees of freedom of the system, i.e., we obtain a smaller effective system. Second, for those spins, which are nearest neighbors of the frozen spins, their local random fields are effectively shifted: $h_i' = h_i \pm J n_i^f$. Here n_i^f is the number of spin i's frozen nearest neighbors and "+" ("−") represents freezing up (down) spins. For the smaller effective system, we can do the mapping again and use the push-relabel algorithm to find the GS.

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